

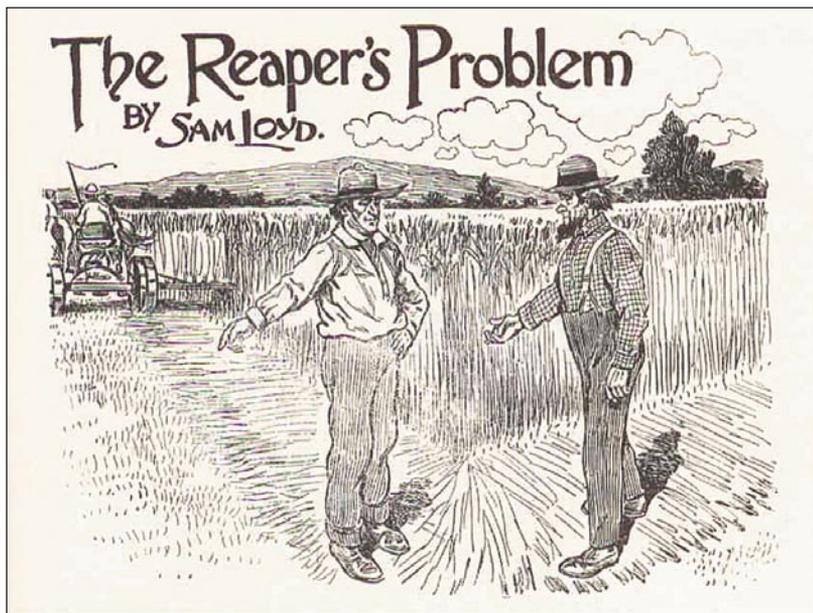
# GENERATING

## PROBLEMS FROM PROBLEMS

# AND

## SOLUTIONS FROM SOLUTIONS

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**M**any mathematical problems can be completely solved analytically. Usually, when we reach a solution, we tend to be satisfied and consider the task finished. In this article, we illustrate the practice of reexamining the result of a full analytic solution, an approach that leads to the search for an alternative geometrical solution and thus enhances our understanding of both solutions.

### THE REAPER'S PROBLEM

The Reaper's problem, devised by Sam Loyd (Gardner 1960), provides a good illustration of this practice:

A Texas ranchman, who owned more land than he could conveniently farm, leased half of a certain field to a neighbor. This field was 2,000 yards long by 1,000 yards wide, but because of certain bad streaks

which ran through the land it was decided that a fairer division would be obtained by cutting a band around the field....

I presume our puzzlists will find no great difficulty in determining the width of a border strip, to be cut all around that field, that will contain exactly half of the total crop. (p. 63)

### INITIAL SOLUTION

The Reaper's problem can be visualized as the rectangle with the given dimensions and a band around it (see **fig. 1**). We know that the outer band and the internal rectangle have the same area. The problem can be modeled by a simple quadratic equation:  $(2000 - 2x)(1000 - 2x) = 1,000,000$  where  $(2000 - 2x)$  represents the length of the inner rectangle and  $(1000 - 2x)$  represents its width. Thus, the area of the inner rectangle should be half the area of the larger rectangle.

The problem can be also modeled by the following equation:  $2x(1000 - 2x) + 2x \cdot 2000 = (2000 - 2x)(1000 - 2x)$ , where the right side is the area of the outer band and the left side is the area of the inner rectangle. There may be other ways to model this problem.

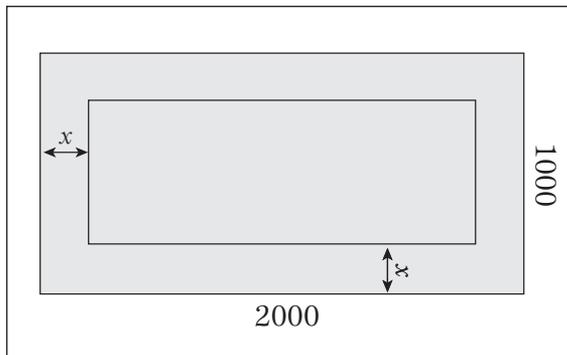
In any case, the solution of these equations yields two results for the width of the outer band: approximately 1309.016 and 190.983. Only the latter fits the conditions of the problem. Thus, the Reaper's problem is more an exercise than a problem, because it requires only the translation of the situation into an equation whose solution is straightforward.

### A FIRST ATTEMPT AT GENERALIZATION

The statement of the Reaper's problem concludes as follows: "There is a simple rule which will apply to *any* rectangular field" (Gardner 1960, p. 63; emphasis added), a statement that invites a generalization. Generalizing is what many experienced problem solvers often do after reaching a particular numerical solution. In this case, a generalization would consist of solving the problem for a general rectangle having sides  $a$  and  $b$ . For example, we can solve  $(a - 2x)(b - 2x) = (1/2)ab$  to obtain

$$x = \frac{a + b \pm \sqrt{a^2 + b^2}}{4}.$$

Simple algebraic or geometrical considerations show that, of these two possible solutions, only the version with the negative square root is reasonable. The other version of the numerator yields an impossible value for the width of the outer band, which turns out to be larger than the sum of the sides of the original rectangle. Further, this difference is always positive.



**Fig. 1** One way to divide the farmland in half

### WHEN THE SOLUTION BECOMES THE PROBLEM

The solution to Loyd's problem states:

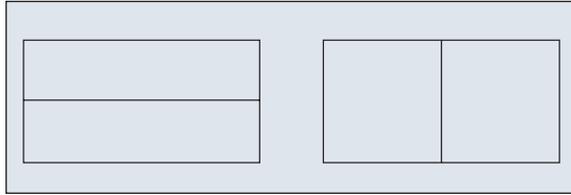
In that interesting problem of the reapers who cut a swath around a rectangular field until half the crop was gathered, I find that they had a simple rule. They said: "One quarter the difference between a short cut cross lots, and round by the road." Mathematicians will understand it better if we say: from the sum of the two sides subtract the diagonal of the field and divide the remainder by four. The field was 2,000 yards long by 1,000 yards wide. Using a tape line, those honest farmers found that the diagonal from one corner to the opposite one was little over 2,236 yards. To "go round by the road," of course, was 3,000 yards, so the difference was a little less than 764 yards. One quarter of this is just a bit shy of 191 yards (190.983), which is the width the border strip should be. (Gardner 1960, p. 49)

We did not spontaneously attribute geometrical meaning to the general solution

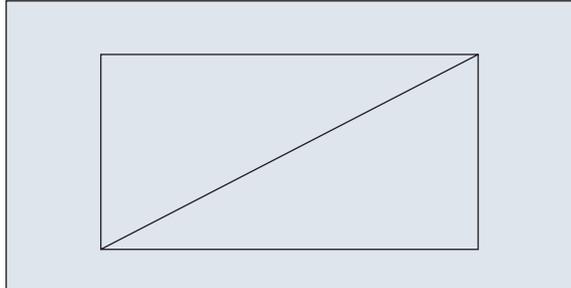
$$x = \frac{a + b - \sqrt{a^2 + b^2}}{4}.$$

However, this expression can certainly speak to us in geometrical language: From the sum of the sides, subtract the length of the diagonal and divide the result by 4.

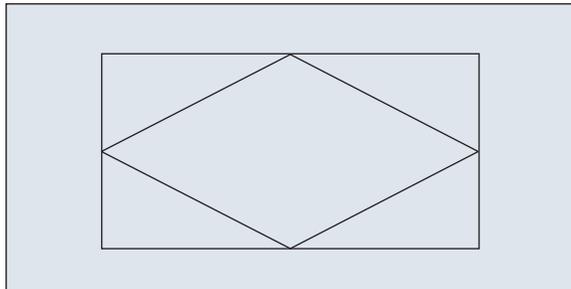
By interpreting Gardner's rule in light of an analytic solution, we sensed that the solution was becoming a new problem on its own. Once we interpreted the result geometrically, we felt that, beyond merely imposing a geometrical interpretation on the analytic solution, we needed to make geometrical sense of it. In other words, the new problem becomes this: Can we produce a visual or geometrical *justification* (not just an interpretation) of the rule that states that one-fourth of the difference between the sum of the sides and the diagonal is the width of a strip dividing the original rectangle in half? In other words, can we find a way to use diagrams, lengths, and areas (instead of symbols)



**Fig. 2** Two simple ways to divide a rectangle in half



**Fig. 3** Using the diagonal to divide the rectangle in half



**Fig. 4** Another possible way to divide a rectangle in half

to justify the rule geometrically? If we found such a geometrical justification, it would add explanatory power to the impeccable but rather opaque analytic solution. We invite readers to pause and think about a geometrical justification of the rule.

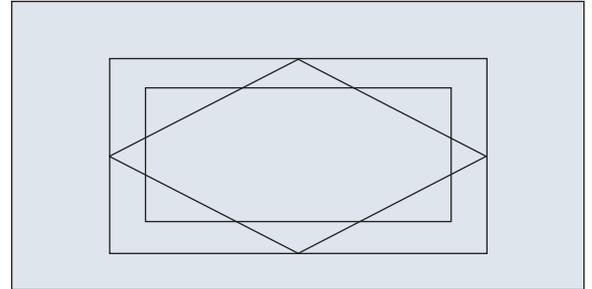
### A POSSIBLE SOLUTION TO THE NEW PROBLEM

What might be the inspiration for such a geometrical solution? An obvious heuristic in this case would be to start thinking geometrically. For example, what are the possible ways we could divide the rectangle into two equal parts, and how would they compare with the way we are interested in? The most obvious is a line joining midpoints of the length or the width (see **fig. 2**).

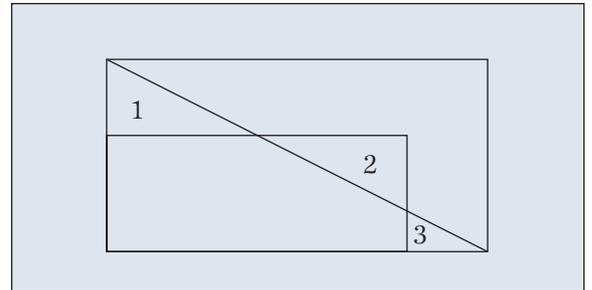
But there is not much to learn from comparing this way of dividing the rectangle with the way suggested in the problem. Given the statement we are trying to make geometrical sense of—“one-fourth of the difference between the sum of the sides and the diagonal”—perhaps we should look for ways of dividing the rectangle that involve the diagonal (see **fig. 3**).

But, again, no progress. Would other ways that divide the rectangle into two equal parts and that involve the diagonal be more insightful?

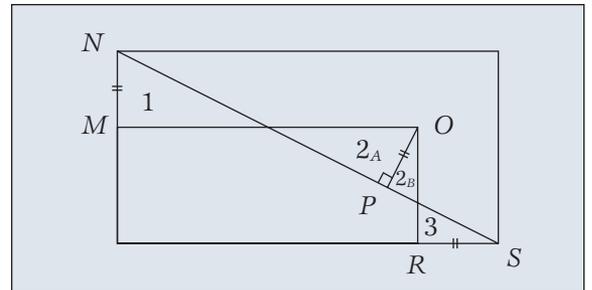
The following came to mind: The area enclosed



**Fig. 5** Superposition of two ways of dividing the rectangle in half



**Fig. 6** The upper-right corner of **figure 5**



**Fig. 7** Comparison of the areas of the three small triangles

by the rhombus is one-half the area of the rectangle, and each of its sides is half the diagonal (see **fig. 4**). Thus, if we consider the four congruent triangles as a single unit, we have indeed divided the rectangle into halves. The number 4 may be related somehow to dividing by 4 in the formula above, so this approach may have some promise. A careful examination of the superposition of two ways of dividing the rectangle (see **fig. 5**) helped us make progress.

Because the four corners are congruent, we focus on one of them for the moment. If we want the bordering strip to be half the area, the combined area of triangles 1 and 3 should be equal to the area of triangle 2 (see **fig. 6**). In triangle 2, we draw the altitude  $OP$  (see **fig. 7**). It can be easily proven that triangles 1 and  $2_A$  and triangles  $2_B$  and 3 are congruent if and only if  $MN = OP = RS$ . Both pairs of triangles are right angled and have vertical angles. If the equality  $MN = OP = RS$  holds, the triangles are congruent by the SAA theorem, in which case the areas compensate for each other and the whole strip is indeed half the original rectangle. If  $MN = OP = RS$  holds, then **figure 7** can also be seen as highlighting the equal segments (see **fig. 8**).

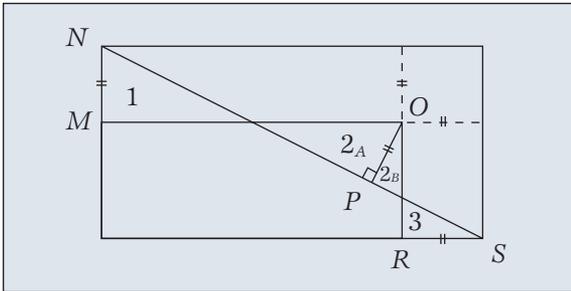


Fig. 8 Highlighting the equal segments

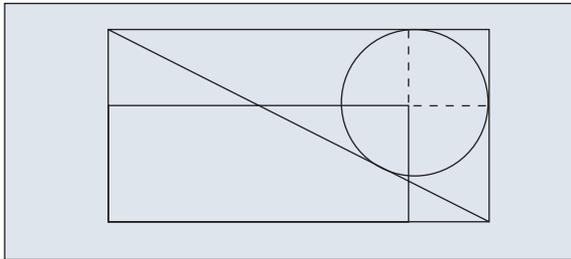


Fig. 9 Inscribing a circle

Again, we invite the reader to pause and think. What might these three equal segments suggest? To put it another way, what might the fact that  $O$  is equidistant to the sides of the rectangle suggest? The three equal segments from  $O$  suggest the possibility of inscribing a circle in the larger triangle, as shown in **figure 9**.

Let us reposition the triangle for visual convenience (see **fig. 10**). This diagram suggests that we dissect the triangle (see **fig. 11**). The figure makes some relationships between the lengths involved become more apparent. Because of the congruence of the small triangles, we observe that the sum of the lengths  $d - r$  and  $e - r$  equals the length of  $f$ , or  $d - r + e - r = f$ . Thus,  $r = (e + d - f)/2$ .

But  $d$  is half one of the sides (side  $a$ ) of the original rectangle,  $e$  is half the other side (side  $b$ ),  $f$  is half the diagonal

$$\frac{\sqrt{a^2 + b^2}}{2},$$

and  $r$  is the width sought. Thus, by making the substitutions in  $r = (e + d - f)/2$ , we obtain the width we are looking for:

$$x = \frac{\frac{a}{2} + \frac{b}{2} - \frac{\sqrt{a^2 + b^2}}{2}}{2}$$

In other words, the width of the outer band is one-fourth the difference between the sum of the sides and the diagonal, originally expressed as

$$x = \frac{a + b - \sqrt{a^2 + b^2}}{4}.$$

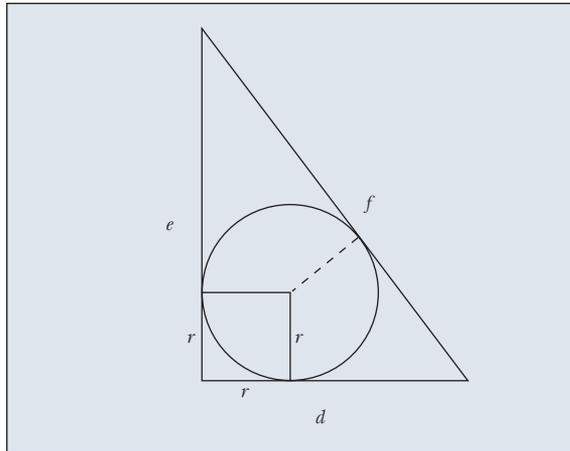


Fig. 10 The triangle with the inscribed circle

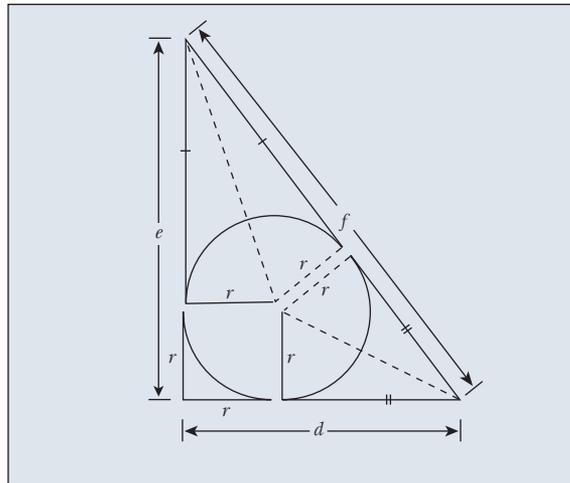


Fig. 11 Dissection of the triangle with the inscribed circle

## POSTSCRIPT

If we examine the analytic solution to a problem in a meaningful way, we may go well beyond merely confirming the correctness of the procedures and forward to posing a new problem. In this case, the new problem is not a related one, because the problem situation remained exactly the same. However, the purpose of posing the new problem was to look for, first, a geometrical interpretation and, second, a geometrical justification. Mathematics educators claim that such “problem posing can help you gain deeper understanding of a standard topic and can enable you to see it in a new light” (Brown and Walter 1983, p. 1).

For some problems, analytic solutions are long and tedious. Such problems certainly invite a search for alternative (possibly visual) solutions. Consider, for example, the following problem:

For what values of  $a$  does the pair of equations

$$\begin{aligned} x^2 - y^2 &= 0 \\ (x - a)^2 + y^2 &= 1 \end{aligned}$$

have either 0, 1, 2, 3, 4, 5, 6, 7, or 8 solutions?

(Arcavi 1994, p. 26)

Problems stated in algebraic terms, like this one, quite strongly invite an algebraic solution. Therefore, it is not surprising that many students approach the problem procedurally, without (or despite) realizing that the algebra involved could be quite laborious and replete with opportunity for error. The Cartesian plane and a geometrical interpretation suggest another way to look at this problem: We look for the number of intersections between the two diagonals of the Cartesian plane ( $x^2 - y^2 = 0$ , or  $y = \pm x$ ) and a family of circles of radius 1 whose centers lie on the  $x$ -axis. From here, the solution emerges rather easily.

However, we would like to claim that even for problems having elegant and efficient analytic solutions (such as the Reaper's problem), we can embark on a quest for further meaning and sense making after we have solved the problem. We suggest that a reexamination of clean analytic solutions may, rather than conveying a message of distrust of symbols, add new layers of meaning.

Analytic solutions, although powerful and general, may remain opaque for many students (who may not even realize that they remain opaque). Many students are capable of performing well with procedures, reaching a solution to a problem, and feeling satisfied. However, they may not develop sense-making habits and a "feel" for symbols—that is, "symbol sense" (Arcavi 1994). A feel for symbols and what they represent can be nurtured by continuously relating analytic solutions to alternative solutions. Visual and other meaning-making strategies have the potential to engage students' understandings far beyond the symbolic.

We invite readers—teachers and students—to develop the healthy habit of searching for alternatives to analytic solutions. Whenever we "finish" a problem, we will pose the following questions: "What does this solution mean, verbally and geometrically?" and "Can I use another representation to relate the symbols to the meaning of the situation they represent?" This process may take time, and students working independently may not be able to develop an alternative solution like the one presented here. But guiding students

to search for alternative solutions and reflect on the value of the search is an important component of helping them become mathematical thinkers.

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