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COMPUTER MEDIATED LEARNING: AN EXAMPLE OF AN APPROACH

ABSTRACT. There are several possible approaches in which dynamic computerized environments play a significant, and possibly unique, role in supporting innovative learning trajectories in mathematics in general, and particularly in geometry. These approaches are influenced by the way one views mathematics and mathematical activity.

In this paper we briefly describe an approach based on a problem situation and our experiences using it with students and teachers. This leads naturally to a discussion of some of the ways in which parts of the mathematics curriculum, classroom practice, and student learning may differ from the traditional approach.

KEY WORDS: dynamic geometry, representations, functions, explanations, hypothesizing.

1. RATIONALE

Dynamic computerized environments constitute virtual labs in which students can play, investigate and learn mathematics. The following are some of the characteristics in which such labs have the potential to nurture, provided they are accompanied by suitable curriculum materials and classroom practices.

1.1. *Visualization*

"Visualization generally refers to the ability to represent, transform, generate, communicate, document, and reflect on visual information" (Hershkowitz, 1989, p. 75). As such it is a crucial component of learning geometrical concepts. Moreover, a visual image, by virtue of its concreteness, "is an essential factor for creating the feeling of self-evidence and immediacy" (Fischbein, 1987, p. 101). Therefore, visualization "not only organizes data at hand in meaningful structures, but it is also an important factor guiding the analytical development of a solution." (ibid.) Dynamic environments not only enable students to construct figures with certain properties and thus visualize them, but also allow the user to transform those constructions in real time. This dynamism may contribute towards

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forming the habit of transforming (either mentally or by means of a tool) a particular instance, in order to study variations, visually suggest invariants, and possibly provide the intuitive basis for formal justifications of conjectures and propositions.

1.2. *Experimentation*

Besides visualization, playing with dynamic environments allows students to learn to experiment, and "to appreciate the ease of getting many examples . . . , to look for extreme cases, negative examples and non stereotypic evidence . . . " (Yerushalmy, 1993, p. 82). They can do so not only by looking, but also by measuring, comparing, changing (or even distorting) figures and making supporting constructions rather easily. The information obtained in this way can be a step towards stating generalizations and conjectures, which in turn should serve as the basis for the next important characteristic.

1.3. *Surprise*

It is unlikely that students will direct their own experimentation fruitfully from the outset. Curriculum activities, such as problem situations, should be designed in such a way that the kinds of questions students are asked can play significant roles in the depth and intensity of a learning experience. One significant type of question to accompany the experimentation, is to require students to make explicit and thoughtful predictions about the outcome of a certain phenomenon or action they are about to undertake. Making such predictions explicitly (a) nudges students to be clearer about how they envision the situation they are working on, (b) brings students to the position of "prediction owners" and thus they are likely to be more careful in the way they think about it, and as a consequence, more committed to the situation, and (c) create expectation and motivation for the actual experimentation. The challenge is to find situations in which the outcome of the activity is unexpected or counter-intuitive, such that the surprise (or puzzlement) generated creates a clear disparity with explicitly stated predictions. (Students working on such activities are described in Hadas and Hershkowitz, 1998, 1999). This can be the trigger for nurturing the students' own need for re-inspection of their knowledge and assumptions, establishing opportunities for meaningful learning.

1.4. *Feedback*

Surprises of the kind described above arise from a disparity between an explicit expectation from a certain action and the outcome of that action.

The feedback is provided by the environment itself, which re-acted as it was requested to do. It is the "dry" consequences of the student action that are to be confronted. Such direct feedback is potentially more effective than the one provided by a teacher, not only because of its affective underpinnings (lack of value judgment), but also because it may engage motivation to re-check, revise the prediction and induce the need for proof. The assumption is that there exists appropriate previous knowledge to appreciate such feedback as meaningful, and that this would serve as the basis for reflection. Moreover, we maintain that the problem situations to be designed for computerized environments (an example of which we propose in this paper) and the teacher implementation of them, should support students to appreciate and formulate "conflicts" or inconsistencies and to find ways to resolve them.

1.5. *Need for Proof and Proving*

Dreyfus and Hadas (1996) discuss and exemplify how one can capitalize on such student surprises in order to instill and nurture the need for justification and proof. Following a surprise, many students may require a proof, maybe not explicitly, but by demanding from others or from themselves an answer to their 'why' (or 'why not').

The engineering of successful tasks should take into account something else as well. If possible, the proof, namely the answer to the 'why', should arise from the observations and the re-visions of the experimentation process itself. In other words, the experimentation-feedback-reflection cycle should provide the seeds for the argumentation which helps to explain and prove an assertion. In this way the dynamic environment really supports the 'closing of the circle'.

In the following we analyze a problem situation which is "... sufficiently concrete – 'well connected to what the learner already knows' – as well as interesting and soluble –" (Noss and Hoyles, 1996, p. 67) with the potential to encourage students to generate genuine and spontaneous questions for themselves. The general theoretical perspective which inspired the design of the problem situation is based on the main ideas by Duval (1999, for example). Briefly stated, the theory claims that an essential component of learning mathematics is the coordination of different representations of a given idea or concept. Such a coordination implies manipulations within a certain representation and translation across representations. The problem situation we present and analyze below is just one example of many, which we believe illustrates and substantiates the claims and processes described above. The description includes the problem situation itself, its implementation with a particular dynamic geometry system

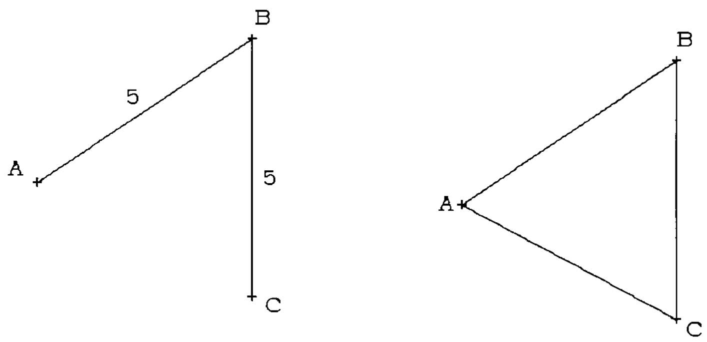

Figure 1. Constructing a dynamic isosceles triangle.

(Geometry Inventor, 1994), and anecdotal data from classrooms (9th and 10th grades, ages 15–16) and from teacher workshops in Israel. The background knowledge of the students (and of the teachers) included: the concept of function in general, symbolic, tabular and graphical (Cartesian) representations of a repertoire of functions, and a substantial amount of Euclidean geometry. The data come from several trials. However, in the following, we neither analyze protocols nor give verbatim quotations. Rather we attempt (a) to describe the flow of the activity and the way it interweaves different ideas across different representations, (b) to propose the intermediate prompts or leading questions (which we refined after several trials) and a sample of interesting responses (not necessarily correct), and (c) to convey the general spirit of the special characteristics of the activity, as an example of the pedagogical and cognitive potential of the use made of computerized technologies. We intersperse comments to analyze and highlight the mathematical, cognitive and pedagogical issues our experiences raise. Finally, we reflect on possible implications.

2. THE PROBLEM SITUATION – FIRST PHASE

The first step consists of building on the "drawing board" (of the Geometry Inventor), two segments of length 5 with a common end point. Joining the two other end points produces an isosceles triangle (Figure 1). The dragging of, for example, the vertex C, yields many possible isosceles triangles whose equal sides are 5 (as in Figure 2).

The effect of the "continuous" and dynamic variation sets the scene for our first question: what changes and what stays the same? Students usually pointed to the given equal sides as constant (some even mentioned the sum of the internal angles) and they mentioned the most obvious variable: the third side AC. When required to find more variables, they also mentioned

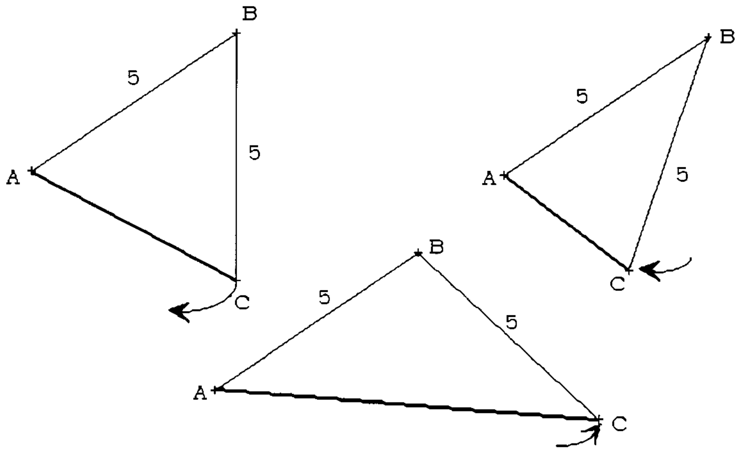

Figure 2. The dynamic change of the triangle.

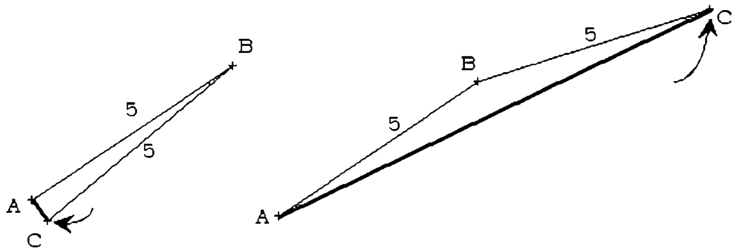

Figure 3. Visualizing the domain of the function.

the angles and the area of the triangle. We proposed to study the variation of the area as a function of AC. By dragging the vertex C, the domain of variation of AC (between 0 and 10) became evident (Figure 3).

Also the range (or co-domain) became evident from the dynamic changes: the minimum area is 0 (when AC is 0), it then increases, but at some point it starts to decrease until it reaches 0 again when AC becomes 10.

Our next question followed quite naturally: predict when the area reaches its maximum value.

One answer consisted of pointing at one or more triangles which look like having a maximum area and saying "the largest triangle is probably one of these". Another answer, more frequent among teachers, was that the equilateral triangle (AC = 5) has the largest area. Given the medium, it is natural that most predictions were geometrical characterizations of the triangle with maximum area, rather than a numerical value for AC or the area. All predictions were recorded (the correct one was not the most

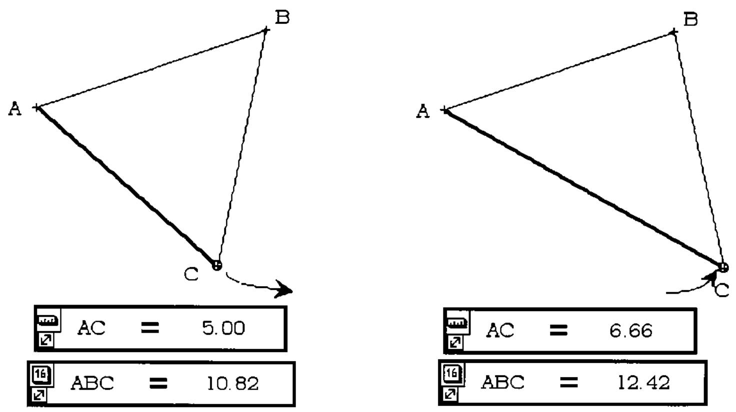

Figure 4. Numeric and figural representations.

common). Then we directed the work back to the "drawing board" for further checks and feedback.

At this stage, we suggested using measurements, which change in real time with the changes in the triangle. Measurements play a significant role in providing useful feedback (and especially counterexamples).

Figures 4(a) and 4(b) illustrate how the screen might look in two discrete shots when measurements are introduced.

Dragging C "beyond" the equilateral triangle (as shown in the transition from Figure 4(a) to 4(b)), usually helps to dismiss the conjecture that it is the one with maximum area: the area values increase beyond 12, towards 12.4, 12.5 and then begin to decrease. The surprise caused by the empirical (numerical) refutation of the "equilateral triangle conjecture" sets the scene for the geometrical work on the 'why'.

Recalling the half-base-times-altitude formula was usually the next step. However, when students or teachers chose AC as the "base", the difficulty was immediate: how to establish the maximum value for the product of two variable quantities (AC and the corresponding altitude)? Thus, someone in the class suggested (and in some classes we did so) changing the perspective and looking for another base-altitude pair in which one of these values remains constant; for example, AB and the corresponding altitude, as shown in Figure 5.

By dragging C again, some were immediately able to "see" not only the correct answer, but also its geometrical justification: the area will be maximum, when the altitude is the largest, namely when DC coincides with BC (Figure 6 illustrates the visually strong dynamic increase of the altitude's size until it reaches its maximum).

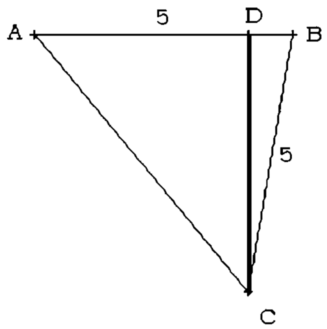

Figure 5. Change of perspective.

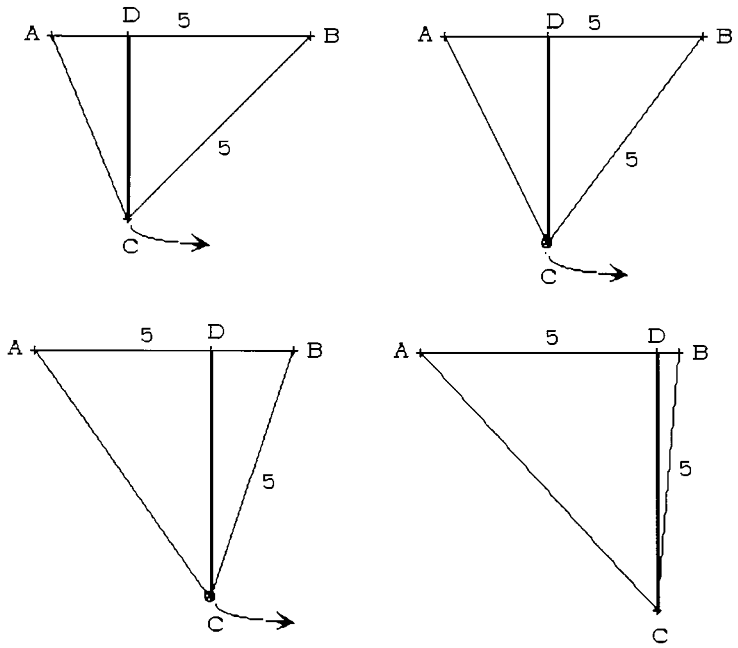

Figure 6. Representing the justification by the dynamic change.

Thus the maximum is reached when ABC is right-angled. This finding is coherent with the 12.5 value for the maximum area (half the base times the altitude is $\frac{5\times 5}{2}$), already observed previously.

To summarize so far, the exploration began with a family of isosceles triangles with equal sides of fixed length changing dynamically, observing variation and making predictions. In most cases, the predictions led to a surprise, which caused the "why". The answer to that why was induced by further experimentation, leading to the right answer and its justification (the variable altitude DC is always a leg of the right-angle triangle BDC

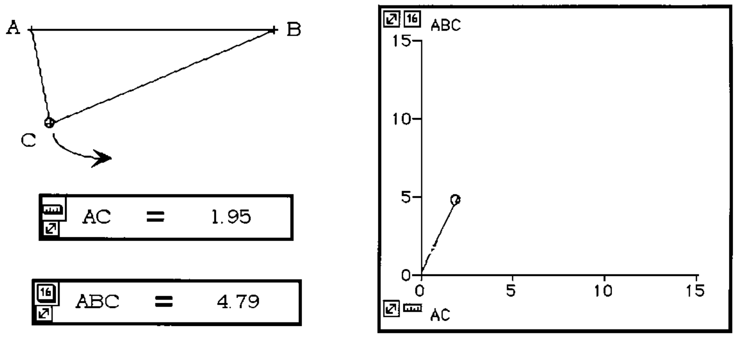

Figure 7. One incipient snapshot of the first graph.

and thus shorter than the hypotenuse of length 5. This is true except when DC coincides with the hypotenuse BC, in which case it reaches the maximum length). Such argument is grounded on a visual-geometrical-dynamic view of the static symbolic formula for the area. Note that the formula was not invoked, nor used, in an operational sense, rather it qualitatively-visually guided the search for the triangle with the maximum area, based on qualitative geometrical arguments.

Usually, at this point, it seemed that the situation had been fully explored, and that there were no more interesting issues to pursue. However, we decided to make use of the of a feature of the software by which one can draw (after matching variables to axes and setting appropriate scales) a Cartesian graph of the variation in real time, as the figure is being dragged. But first, we asked for predictions of the form of the graph of the variation of the area of ABC as AC changes (i.e. as a function of AC). A large majority of participants (students and teachers alike) predicted a parabolic graph, supporting their prediction by saying that the initial area is 0 when AC is 0, it increases to a maximum and then decreases to 0, when AC is 10.

Schwarz and Hershkowitz (1999), report that many students tend to think in terms of two prototypical functions: linear and quadratic. In this particular problem situation, this manifested itself explicitly: a graph which describes an increase from 0 towards a maximum and then back to 0, must be a parabola. Possibly, as we shall see later, there was also an implicit expectation of symmetry as well. In order to check the prediction, we asked students to make use of the software to draw the graph. Figure 7 shows two static snapshots of what happens dynamically on the screen. Dragging C further, Figure 7 turns into Figure 8.

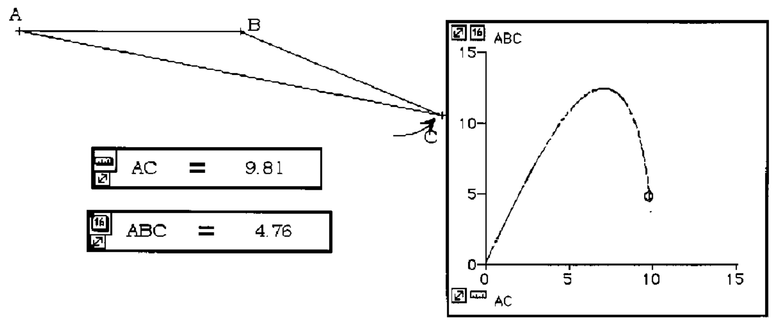

Figure 8. The asymmetry of the graph.

At this point, most were surprised to notice the apparent asymmetry of the graph (with respect to a vertical axis), against their explicit expectation of a parabola, and posed the question "why?" or "why not (symmetrical)?"

In our trials, the realisation (and subsequent explanation) that the maximum area is reached when the triangle is right-angled, was not explicitly connected to the possible shape of a graph (and why should it be?). However, against the visual impact of the asymmetry of the graph and the need to explain it, and (in some cases) following our questions ("what would it mean for the graph to be symmetrical? Where would its maximum be?"), attention was redirected to the values of AC for which that maximum is reached. As a result, some produced the answer: among other things, symmetry of the graph would have implied that the maximum is in the middle of the domain (AC = 5, namely when the triangle is equilateral), but that was already discarded. The maximum was somewhere else – beyond the middle of the domain – when the triangle is right-angled, for which the value of AC is $\sqrt{5^2 + 5^2} \approx 7.07$ (found using Pythagoras).

For some, this explanation ended the matter. For others (especially teachers), in spite of understanding the explanation of why the graph cannot be symmetrical, it did not. They felt and expressed discomfort, because their intuitive feeling of symmetry, which now became explicit, was still strong. Perhaps, it was even reinforced by what they have found about the maximum area being for the right-angled triangle, since this triangle may appear to be "half way" between the extreme values of the domain when one makes the dynamic changes. We thought that we needed to address this issue, even if we had not planned to do so. We therefore suggested for discussion the proposal that maybe there is indeed something symmetrical going on here. The discussion and the reasoning went more or less as follows: for the graph to be symmetrical, it is necessary that the x-value for the maximum area splits the domain of the 'independent variable' into two halves. But, the maximum is obtained for a right-angled triangle, where the right angle is between the two constant sides. That angle varies between 0° and 180°; therefore if the independent variable were to be that angle, the x-value of the maximum is indeed the mid-value of its domain. Thus if we graph the area as a function of the angle between the equal constant sides, it might be symmetrical! An empirical confirmation of symmetry was obtained by taking the angle ABC (instead of the side AC) as the independent variable to draw the graph of the area (as a function of the angle). We did not pursue this further to show that the graph corresponds to $y = 12.5\sin x$, which although it is not a parabola, is indeed symmetrical.

In this case, we encouraged the examination of the graphical representation to enable the re-viewing of the phenomenon. This led to an awareness of properties which went unnoticed for many (or even counter-intuitive for some) when merely observing the phenomenon itself. Moreover, the realisation of these properties could be taken as a trigger to re-analyze the sources of a strong intuition (of some), which was shown to be "wrong" in some sense and "right" in another.

So far, the exploration of the situation, which took advantage of the special features of the computerized environment, was based on

(1) the dynamic manipulation of the situation itself (i.e., dragging one vertex of the triangle and noticing/conjecturing properties);
(2) measurements (which change in real time as the triangle changes and thus enable quantification of the visual phenomenon observed);
(3) the graph (interpreting the graph as another descriptor of the situation helped to identify properties previously unnoticed); and
(4) questions, discussions and reflection, based to a large extent on making sense of outcomes which were at odds with those predicted.

"Absent" from the activity until now, was the symbolic representation of the function describing the variation of the area of ABC as a function of the variable side AC. The symbolic representation of the problem is certainly compact, precise and general ($A = \frac{x\sqrt{100-x^2}}{4}$, where x is the length of AC and A the area), and it encapsulates all the information we found above (domain, co-domain, asymmetry etc.). However, if this problem were to be investigated by first creating the symbolic representation, we would have distanced ourselves from the phenomenon itself, by trying to decode the somehow cryptic symbols.

Consider, for example, finding the maximum symbolically. Firstly, it may imply knowledge and proficiency in algebraic techniques (possibly calculus). Secondly, such a treatment directs attention and mental energy to rather syntactic issues, relegating or "forgetting" for a while the reference situation, and in the best of cases, postponing meaning until a result is

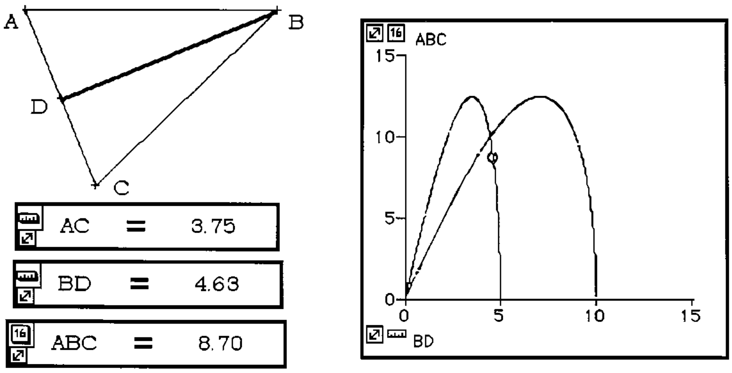

Figure 9. The two graphs in juxtaposition.

obtained. Such a trajectory does not always allow for raising and discussing 'why' questions. Thirdly, the symbolic solution, by its very static and general nature, seems to engulf, and thus dissociate itself from, the visual and dynamic views of the problem and the nuances thereof as described above.

In our trials, we introduced the symbolic representation *after* the investigation (described above) took place. In the spirit of developing "symbol sense" (Arcavi, 1994), the emphasis then became not on learning about the situation from the symbols, but mostly to decode and trace how the information, we already know and expect, is expressed in symbols. As we shall see below, in the next phase of the activity, symbols, when brought at the end of the exploration, can also play a further, more explanatory, role.

The area of the triangle was explored as a function of AC (the variable side) and we also made reference to the area as a function of the angle (between the two equal sides). Therefore, it was natural to propose exploration of the area of the triangle, this time as a function of yet another variable: its altitude to the variable side (from B to AC). What would the graph look like, symmetrical or asymmetrical?

Playing with the software and looking at the measurements, suggested to many that the graph may look similar to the graph of the area as a function of the base. We proposed drawing the two graphs in juxtaposition, for which the screen looks like Figure 9.[1]

Some asked why is the graph of the area as a function of the altitude "thinner" than the other? And we added other issues to consider such as: the graphs appear to reach the "same height" – do they? If so, why, otherwise, why not? What is the meaning, in terms of the geometrical situation, of the point of intersection of the two graphs?

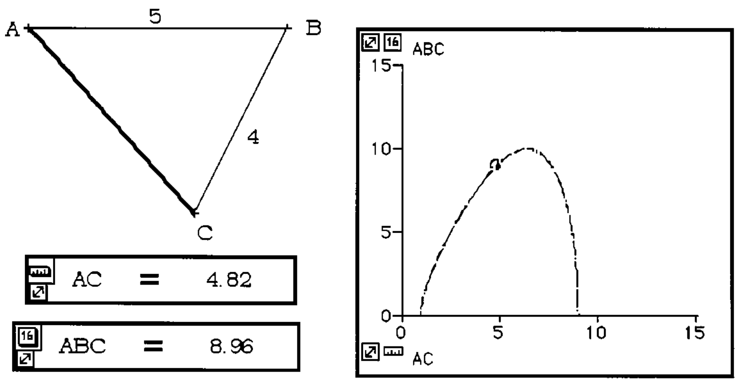

Figure 10. The area as a function of the variable side.

Again, the observation of a certain representation, in this case the graph, posed questions whose answers should be looked for in the phenomenon. We skip here the details of the discussion of these questions, in order to focus on the flow of the problem situation towards what we believe was its “climax”.

3. THE PROBLEM SITUATION – SECOND PHASE

Following again the “what if?” approach, the question became: “So far, we explored isosceles triangles where the equal sides have a fixed given value. What would happen if we make a ‘small’ change, so that the triangle is not isosceles, but “close” to being one?”

The software allows for changes which propagate to the whole construction to be made easily. Thus, using this facility, we proposed changing only one of the fixed sides of length 5 to length 4 (leaving the other with length 5). Exploring how the area varies as a function of the variable side leads to the graph in Figure 10.

The small surprise which we explored here was that the graph “does not start” at the origin. However, the greatest surprise came in the next step. Here again we proposed to investigate the area as a function of the altitude to the variable side. The following (Figure 11) is a series of snapshots of the graph as it is drawn by the computer when the vertex C is dragged (when starting with triangles for which the altitude falls inside).

The incipient graph seemed to be very similar to those previously obtained. In some classes, we did the dragging in front of the class. The general expectation was for a similar shape to those previously obtained (as in Figure 9). However, further dragging yields what we see in Figure 12.[2]

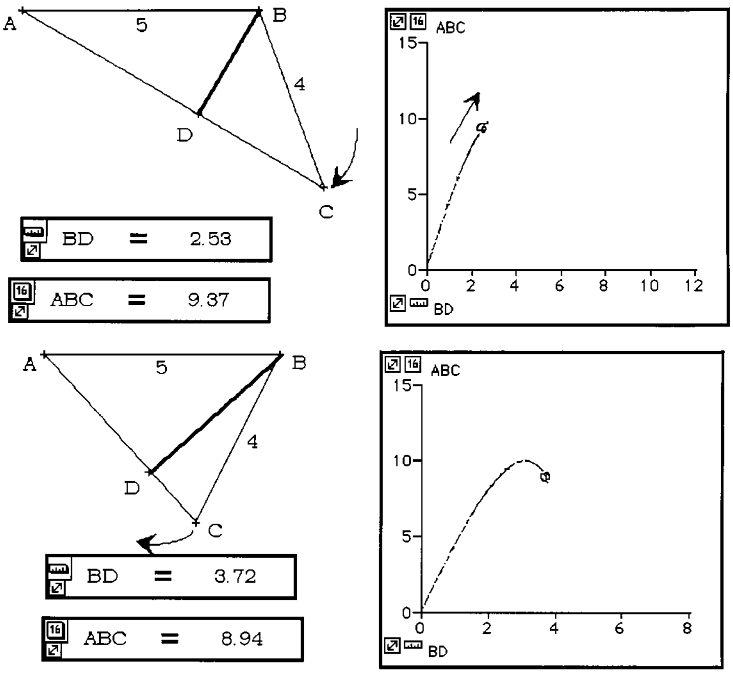

Figure 11. Incipient snapshots of the area as a function of the altitude.

The great majority of teachers were more surprised than the students, and the most frequent spontaneous exclamation we heard was: "... but it is not a function!" For many teachers, the visual "vertical line test" to determine whether a graph is a function or not is a tool much used and invoked, and graphs which fail the test are somehow suspect (see, for example, Even and Bruckheimer, 1998). For others, the graph "going backwards" was the most surprising feature.

The immediate question (which we did not need to raise) was "what is going on here?"

The first step many participants took was to replay the situation, and to notice that, a certain stage, the altitude "escapes" outside the triangle (or jumps from outside to inside the triangle). We took the opportunity to discuss why this had not happened in the isosceles triangle. The altitude escaping outside was suspected to be somehow linked to the surprising graph type. Others linked that observation to their further noticing that the altitude does not increase in all its domain, it also "shrinks" in some parts of it, which explains why the graph "goes backwards". Others also observed that the increasing and then decreasing values of the altitude (the independent variable) creates two different values of the area for the same

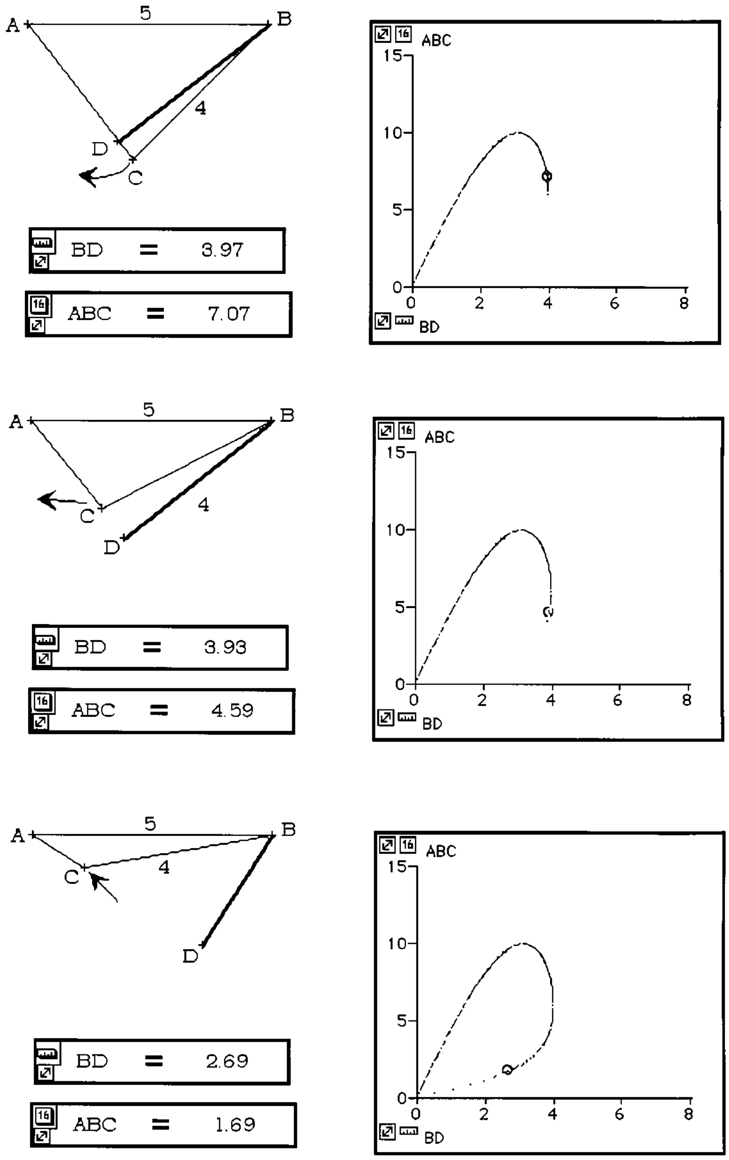

Figure 12. The surprising graph.

value of the altitude: if the altitude is outside the triangle the 'base' is smaller and thus the area is smaller, but the same altitude can be inside the triangle, and this happens for a larger 'base', in which case the value for the area is larger (see Figure 13). We found these kinds of explanation especially interesting because: (a) they emerged after playing and replaying the geometrical situation dynamically, (b) they are closely related to

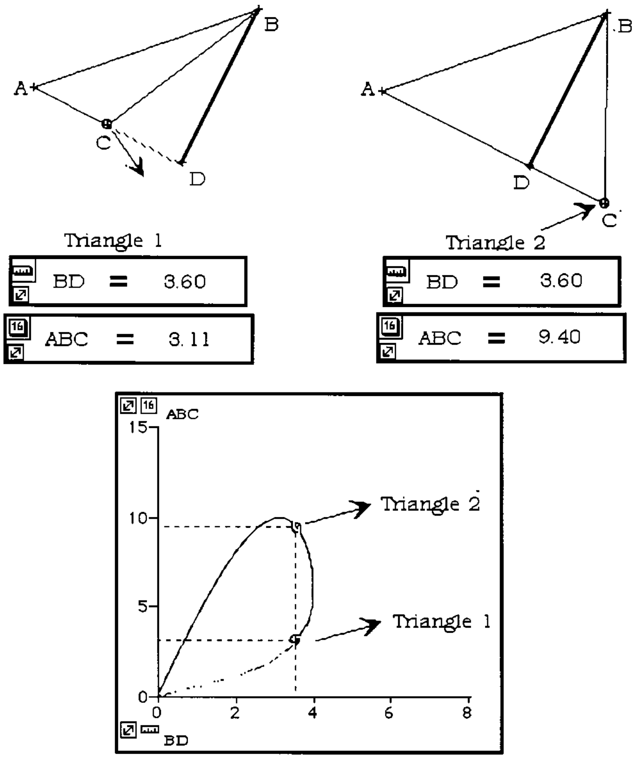

Figure 13. Figural explanation of the surprise.

perceptually experiencing the variability which is at the core of the idea of a function as depicting a dynamic situation, and (c) although they emerge from empirical observations, they serve as stepping stones to produce more general, deductive arguments which seemed to be pleasantly convincing.

At this point, we decided to devote some time to the creation and analysis of the symbolic representation, to 'replay' with teachers and students what happened to us when we invented and played with this situation. We asked (as we asked ourselves then): "how would symbols depict this situation?"

For those who produced/followed the qualitative geometrical explanation, the process of modeling this situation symbolically added another layer of meaning. Firstly, it formalized the verbal explanation into $A = \frac{BD \cdot AC}{2} = \frac{BD(AD \pm DC)}{2}$. This formula helped to re-view in symbols how for the same altitude BD, one gets two different values for the area (which was the graphical surprise): once, the area is calculated as the sum of two triangles with this altitude as a common side, and again, when the altitude falls outside, the area is calculated as the difference between two such triangles (see Figure 13).

Secondly, by re-writing the formula in terms of the variable altitude (x = BD) to get A = $\frac{x(\sqrt{25-x^2}\pm\sqrt{16-x^2})}{2}$, it became clearer that, in fact, what we have is two functions depicting this situation, for different subdomains. The graphs of these two functions 'smoothly join' when the altitude coincides with the side of length 4 (which is precisely the maximum value of the altitude). The substitution of that value in the expression provides a further realization that, in this case, there is no sum or difference of two triangles, which were already seen graphically as "collapsing" into a single triangle.

The issues with which we dealt here consisted in comparing, contrasting and moving back and forth from the geometrical situation, its graphical representation and the symbolic expressions. For example, why is 4 the maximum value for the altitude to AC and how do the situation, the graph and the symbols express this? Or, what is the geometrical meaning of the functions joining 'smoothly'? Or, what does it mean that the graph has always (with two exceptions) two x values for each y value and vice versa? Or, what would happen if we were to consider, for example, the area of ABC as a function of the altitude to AB (the side of length 5)? And so on.

We see this activity in a similar way in which Dennis and Confrey (1996, p. 35) describe their views of evolution of knowledge in the history of mathematics: "as the coordination and contrast of multiple forms of representation. …often one sees a particular form of representation as primary for the exploration, whereas another may form the basis of comparison for deciding if the outcome is correct. The confirming representation should be relatively independent from or contrasting the primary exploratory representation. It must show contrast as well as coordination for the insight to be compelling." They claim that, the sense for certain representations is lost and distorted in most curricula. For example, "the Cartesian plane is treated predominantly as a means of displaying algebraic equations". Instead, in our case, the Cartesian graph was directly expressing characteristics of the phenomenon which required explanations. Some of those explanations were produced verbally by re-viewing the phenomenon. These and others were refined and re-produced when the symbolic representation was introduced to contrast, verify and extend the insights.

4. REFLECTION

The problem we proposed and its exploration is a case in point which raises the following issues.

(1) The role of computerized tools
The existence of the computer poses to mathematics educators the challenge of designing activities which take advantage of those features with the potential to support new ways of learning. In the specific case we report here, it was the possibility to represent graphically and dynamically in real time the variation of a geometrical phenomenon which seemed promising.

We played for a while until we came up with a collection of problem situations similar to the one described (Arcavi and Hadas, 1999). Thus, in our case, the trigger to design exploration projects was the availability of the tool and the advantage to be gained from its capabilities. We sensed that we have a powerful means to promote student learning via mathematical investigations and meaning making, which in our view is strongly linked to producing explanations to unexpected phenomena, relying on the phenomena themselves and different mathematical representations thereof.

For us, the encounter with the tool preceded our realisation of its need. On the contrary, it was a starting point to inspire us to harness it in consonance with our views of learning, and thus searching for and designing problem situations accordingly.

(2) Mathematics and mathematical activity
We believe that the problem situation described above and the way we implemented it can be considered as an example of doing some mathematics in a rather new way, instead of using the tool to put “old approaches” into just another dress.

Imagine this situation presented in a more traditional paper and pencil setting; e.g., “Which of the isosceles triangles whose equal sides are of length 5 has the largest area?” Or even in a more open way: “Investigate the area of isosceles triangles with equal sides of length 5.” It is almost certain that symbols would be the first, if not the only, medium to be invoked (possibly with some sketchy drawings to support the modeling process). In our experiences, the graph was produced and used *before* its algebraic representation. Both the situation and its graph are looked at dynamically, all information gathered is intimately related and expressed in terms of the original situation, and is put at the service of better understanding it. It is in the translation between the graph and the situation that more subtleties of the phenomenon unfold. The initial absence of the algebraic representation, does not seem to impede genuine and deep mathematical reasoning.

Quite the contrary, we would claim that its absence helps to keep in mind almost all the time the phenomenon modeled by the graphs, and prevents us from being distracted by symbol manipulation which may distance us from the original meanings. But, when symbols are finally introduced, the "algebraic expression comes alive ..." (Noss and Hoyles, 1996, p. 245), as it is inspected for the ways in which (a) it expresses the information already found, and (b) it may add insight to the analysis (as in the last phase above).

Thus, the activity illustrates new ways of doing mathematics, since:

– Empirical explorations of geometrical phenomena take the form of looking at many particular cases and the dynamic transitions among them. Observation may not only help to reveal patterns, it may also be the source of insight and meaning, and serve as the basis for proving and further exploration.
– Making sense of a geometrical situation while playing with the situation itself first, and then by interpreting its representations (graphs, symbols), seems to enhance both the understanding of the situation and of the representations. The concept of function, which models the geometrical situation, recovers its dynamism, as a genuine model for change and variation since its graphical representation is being created in real time describing the phenomenon as it occurs. Moreover, the interplay between the global (the graph as a whole, the formula as an object) and local views of a function (pointwise view of the function as a procedural connection between two quantities) becomes apparent. Thus, graphs and functions are used both as objects and processes (e.g., Moschkovich et al., 1993) with which to think about aspects of the situation being mathematized.
– The teacher, who brings this problem situation to the classroom with students or with fellow teachers, becomes a guide who poses the appropriate questions at the appropriate moments. For example, the teacher

  (a) requests predictions which encourage students to take a stance on the problem and serve as a background against which to deal with unexpected results (e.g. "Predict when the area reaches its maximum", "Predict the form of the graph");
  (b) requests students to be explicit about the why (or why not) of what they see (e.g. "Why did the graph turn out to be not symmetrical");
  (c) helps to make explicit and to deal with intuitions or knowledge which may underlie an 'incorrect' prediction (e.g. "What

would it mean for the graph to be symmetrical? Where would its maximum be?"),

(d) leads the discussion, poses new questions, and promotes the coordination between different representations.

In other words, the technological tool in itself is of little value if it is not accompanied by problem situations which make meaningful use of it. However, any curriculum also requires the skillful implementation of the teacher roles described above in order to be faithful to its spirit. In our case, it implies engaging student knowledge, making it explicit, building on it, and helping students coordinate seemingly disconnected pieces and conciliate apparent inconsistencies.

– In tasks as the above, the boundaries between mathematical subdisciplines get blurred: functions depict geometrical phenomena, geometric explanations emerge to explain features of the graphical and symbolical representation of the function, and vice versa.

(3) A new way of thinking?

Noss and Hoyles (1996, p. 240 ff.) describe three completely different approaches (produced by a group of mathematics education researchers) to solve the same problem, two of which are "computerized". They report that those who solved the problem were surprised by the "diversity of knowledge, style and solution" involved in each approach. They claim that the choice of medium to solve a problem "mediate the range of meanings and connections which are likely to structure the interaction, and which are likely to emanate from it" (p. 245). Such choice depends upon familiarity, suitability and expertise developed over years of working in a medium, which gradually determine a preferred approach to problems.

In retrospect, we see that that is precisely what we have experienced. Our own continued practice in a certain computerized environment, playing with situations as the one described above, resulted in the development of an approach with which we began to envision a whole class of problems. We found ourselves using Cartesian graphs to represent certain geometrical situations as a way to inspire proofs and to gain insights. This way of thinking was further reinforced during the search for and design of geometrical problem situations in the process of creating the collection we developed.

The rationale for the design was based on what we believe the development of sound mathematical understandings is. But, more subtly, we realise that we are advancing a certain approach, a certain viewpoint, which we developed after having many experiences with it, and which we happen to like.

Although an important component of education is about intervention and about establishing a set of values to influence points of views and practices, several questions remain open. Are we making rational use of the decision power we, as teachers or designers, have to advance a certain practice over others? Or, should we advance several approaches and let students choose? Could it not be that the new ways seem exciting and meaningful to us, designers, because we can enjoy them against the background of what we already know, but could be less fruitful for many students in the long run?

We propose these issues for feedback and reflection on the basis of the problem we analyzed, and as a basis for further research.

NOTES

[1] Since this is a static picture, it highlights the latest graph drawn: the area as a function of the altitude (the "moving point" is on it and the variable appearing on the x-axis is BD).
[2] Some students or teachers who did it by themselves, generated the graph in the "opposite direction". For them, the final result was no less of a surprise.

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